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A FAMILY OF COMPLEMENTED SUBSPACES IN VMO AND ITS ISOMORPHIC CLASSIFICATION

BY

PAUL F. X. MÜLLER*

Department of Mathematics, J. Kepler University Linz
Altenbergerstrase 69, A-4040 Linz, Austria
e-mail: pfxm@bayou.uni-linz.ac.at

ABSTRACT

In this paper we study families of spaces which are similar in spirit to the Rosenthal class. We let $\mathcal{S}^0$ be the infinite dimensional sequence space where the norm of a given null-sequence (a_I) is given as follows,

$$\|(a_I)\|_{\mathcal{S}^0} = \|\sum x_I a_I h_I\|_{\text{VMO}} + \sup |a_I|.$$

Here (x_I) is a fixed sequence of bounded scalars. We show that these spaces are isomorphic to complemented subspaces of VMO, and classify their isomorphic types as follows: $\mathcal{S}^0$ is isomorphic either to c_0, to $(\sum \text{BMO}_n)_0$, or to VMO. The space $\mathcal{S}^0$ arises as endpoint of the scale $\mathcal{S}^p, 2 \le p < \infty$, where the norm of a sequence (a_I) is given by

$$\|(a_I)\|_{\mathcal{S}^p} = \left\| \sum a_I x_I^{1-2/p} \frac{h_I}{|I|^{1/p}} \right\|_{L_p} + \left(\sum |a_I|^p \right)^{1/p}.$$

The isomorphic types of this class are shown to be L^p and ℓ^p.

* *Current Address:* Institut Mittag Leffler, The Royal Swedish Academy of Sciences, Auravägen 17, SE-18262 Djursholm Sweden; e-mail: muller@ml.kva.se
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1. Introduction

The results of this paper are centered around well-known conjectures expressing dichotomies for complemented subspaces of VMO. Let X be such a complemented subspace. The first conjectured dichotomy states that either X contains a complemented copy of $(\sum \ell^2_n)_0$, or X is isomorphic to c_0. Second, either X contains a complemented copy of $(\sum \mathrm{BMO}_n)_0$, or X is isomorphic to a complemented subspace of $(\sum \ell^2_n)_0$. Equally important is the unsolved problem whether there are infinitely many isomorphically different complemented subspaces of VMO. The known examples include $VMO, \ell^2, (\sum \ell^2)_0$, the above three, $c_0, (\sum \mathrm{BMO}_n)_0, (\sum \ell^2_n)_0$, and their direct sums. The space of all null sequences (a_n) such that

$$\left(\sum w_n a_n^2\right)^{1/2} + \sup |a_n| < \infty$$

is a further example of a Banach space isomorphic to a complemented subspace of VMO. It is isomorphically different from the first group of examples when $\lim w_n = 0$, and $\sum |w_n| = \infty$. We call it then the Rosenthal space.

In this paper we study families of spaces which are similar in spirit to the Rosenthal class. It was believed, for some time, that they might be the key to the construction of many more examples of complemented subspaces in VMO. We let $\mathcal{S}^0$ be the infinite dimensional sequence space where the norm of a given null-sequence (a_I) is given as follows,

$$||(a_I)||_{\mathcal{S}^0} = ||\sum x_I a_I h_I||_{\mathrm{VMO}} + \sup |a_I|.$$

Here (x_I) is a fixed sequence of bounded scalars. We show that these spaces are isomorphic to complemented subspaces of VMO, and we determine their isomorphic type. Specifically, we prove that $\mathcal{S}^0$ is isomorphic either to c_0, to $(\sum \mathrm{BMO}_n)_0$, or to VMO. Thus we provide some support for the above dichotomy conjectures.

The space $\mathcal{S}^0$ arises as endpoint of the scale $\mathcal{S}^p, 2 \le p < \infty$, where the norm of a sequence (a_I) is given by

$$||(a_I)||_{\mathcal{S}^p} = \left\| \sum a_I x_I^{1-2/p} \frac{h_I}{|I|^{1/p}} \right\|_{L_p} + \left(\sum |a_I|^p\right)^{1/p}.$$

Very recently D. Kleper and G. Schechtman showed that $\mathcal{S}^p$ is isomorphic to a complemented subspace of L^p, for $2 \le p < \infty$. The present paper continues this line of investigation, and gives the isomorphic classification for $\mathcal{S}^p$: When $2 \le p < \infty$ the infinite dimensional space $\mathcal{S}^p$ is isomorphic to L^p or ℓ^p. For this classification problem the detour to VMO turned out to be instructive and rewarding. For instance, we read off the isomorphic type of $\mathcal{S}^p$, from a stopping time decomposition $\{\mathcal{H}_K : K \in \mathcal{E}\}$, which is a common tool in the study of VMO. For $K \in \mathcal{E}$, we let $\mathcal{H}_K$ be the **maximal block** of intervals satisfying

$$\sum_{L \in \mathcal{H}_K} x_L^2 h_L^2 \leq 2.$$

Our Theorems 2, 4 and 5 show that the Carleson constant of $\mathcal{E}$,

$$\sup_I \frac{1}{|I|} \sum_{J \subseteq I, J \in \mathcal{E}} |J|,$$

and the measure of $\limsup \mathcal{E}$ are the isomorphic invariants that determine the Banach space structure of the class $\mathcal{S}^p, 2 \leq p \leq \infty$.

2. Complemented subspaces of BMO

In this section we define the representation of the spaces $\mathcal{S}^p$, respectively $\mathcal{S}^\infty$, as subspaces of L^p, respectively BMO. We will also define the stopping time decomposition mentioned in the introduction. These two constructions and their interplay will be analyzed carefully in this and the remaining sections of this paper. We also establish here the BMO estimates for the induced orthogonal projection.

We let $\{h_I\}$ denote the L^∞-normalized Haar basis, indexed by dyadic intervals. For a sequence of scalars (a_I) we say that $\sum a_I h_I$ belongs to BMO if

$$\sup \frac{1}{|J|} \sum_{I \subseteq J} a_I^2 |I| < \infty.$$

For the above expression we write $\|\sum a_I h_I\|_{\mathrm{BMO}}$.

Now fix a sequence of scalars $x_I \in R$, such that $|x_I| < 1$. We define the infinite dimensional sequence spaces $\mathcal{S}^\infty$. The norm of a bounded sequence (a_I) is given by

$$\|(a_I)\|_{\mathcal{S}^\infty} = \left\| \sum x_I a_I h_I \right\|_{\mathrm{BMO}} + \sup |a_I|.$$

The norm depends of course explicitly on the choice of scalars $x_I \in R$ used on the right hand side of the above expression. Nevertheless our notation suppresses this dependence; for the resulting spaces we write simply $\mathcal{S}^\infty$. Incidentally, $\mathcal{S}^\infty$ is a subspace of BMO $\oplus \ell^\infty$. Let $\{e_I\}$ denote the unit vector basis of ℓ^∞, indexed for convenience by dyadic intervals. Then the subspace of BMO $\oplus \ell^\infty$, spanned by the vectors

$$b_I = x_I \cdot h_I \oplus e_I,$$

is clearly isometric to $\mathcal{S}^\infty$. Indeed for any linear combination, $\sum a_I b_I \in$ BMO $\oplus \ell^\infty$, we have the following expression for the norm,

$$\left\| \sum a_I b_I \right\|_{\mathrm{BMO} \oplus \ell^\infty} = \left\| \sum a_I x_I h_I \right\|_{\mathrm{BMO}} + \sup |a_I|.$$

Our first theorem concerning these spaces states that each of them is isomorphic to a complemented subspace of BMO. The norm of embedding and projection are independent of the choice the scalars $x_I \in R$.

THEOREM 1: *For every choice of scalars $x_I \in R^+$ with $|x_I| < 1$, there exists a sequence $\{g_I\}$ which in* BMO *is equivalent to the sequence $\{x_I \cdot h_I \oplus e_I\}$ in the space* BMO $\oplus \ell^\infty$. *Moreover the weak *-closure of* span$\{g_I\}$ *is complemented in* BMO.

Proof: Let $x_I \in R^+$ be a sequence of scalars with $|x_I| < 1$. For convenience we assume that each of the numbers x_I is a negative power of 2. Then choose $m_0 << m_1 << m_2 \cdots$. Let $\mathcal{D}_i$ be the dyadic intervals of length 2^{-m_i}. First define the collection $\mathcal{C}_{[0,1]} = \mathcal{D}_0$. Then select a collection $\mathcal{B}_{[0,1]} \subseteq \mathcal{C}_{[0,1]}$ such that

$$x^2_{[0,1]} = |B_{[0,1]}|,$$

where $B_{[0,1]}$ is the pointset covered by $\mathcal{B}_{[0,1]}$. Denote by I_1 the left half of the unit interval and by I_2 the right half of the unit interval. Next fix $J \in \mathcal{C}_{[0,1]}$, and select disjoint collections of pairwise disjoint dyadic intervals $\mathcal{C}_{I_1}(J) \subset \mathcal{D}_1$ and $\mathcal{C}_{I_2}(J) \subset \mathcal{D}_1$ so that the pointset covered by $\mathcal{C}_{I_i}$ is contained in J and has measure equal to $|J|/2$. Moreover, we select the intervals in such a way that the relative density of $\mathcal{C}_{I_i}(J)$ is the same on the left half of J and on the right half of J, that is,

$$|C_{I_i}(J) \cap J_j| = \frac{1}{2}|J_j|,$$

where $C_{I_i}(J)$ is the pointset covered by $\mathcal{C}_{I_i}(J)$, and where J_1 is the left half of J, and J_2 is the right half of J. Next we select $\mathcal{B}_{I_1}(J) \subseteq \mathcal{C}_{I_1}(J)$ and $\mathcal{B}_{I_2}(J) \subseteq \mathcal{C}_{I_2}(J)$ such that

$$\frac{1}{2}x^2_{I_1} = \frac{|B_{I_1}(J)|}{|J|} \quad \text{and} \quad \frac{1}{2}x^2_{I_2} = \frac{|B_{I_2}(J)|}{|J|},$$

where $B_{I_i}(J)$ is the pointset covered by the collection $\mathcal{B}_{I_i}(J)$. Now we take the union over $J \in \mathcal{C}_{[0,1]}$ obtaining for $i \in \{1,2\}$ the collections

$$\mathcal{C}_{I_i} = \bigcup_{J\in\mathcal{C}_{[0,1]}} \mathcal{C}_{I_i}(J) \quad \text{and} \quad \mathcal{B}_{I_i} = \bigcup_{J\in\mathcal{C}_{[0,1]}} \mathcal{B}_{I_i}(J).$$

We continue by induction following the pattern of the first step. Thus for each I we construct two families, $\mathcal{B}_I$ and $\mathcal{C}_I$, of pairwise disjoint dyadic intervals satisfying the following properties: The collection $\mathcal{B}_I$ is a subcollection of $\mathcal{C}_I$. Let C_J be the pointset covered by the collection $\mathcal{C}_J$. Then $|C_J| = |J|$, and moreover the family $\{C_I\}$ is a nested family of sets, so that $C_I \subseteq C_J$ if $I \subseteq J$. Note that if $K \in \mathcal{C}_I$ and $L \in \mathcal{C}_J$ and $K \subseteq L$ then we have $C_I \subseteq C_J$, or equivalently, $I \subseteq J$. Moreover, we observe the following: Fix a dyadic interval J, and let

$$g_J = \sum\{h_K : K \in \mathcal{B}_J\}.$$

Suppose that I is a dyadic interval which is strictly contained in J. Then for $K \in \mathcal{B}_J$, we have the following identities expressing a strong degree of self-similarity,

$$\frac{1}{|K|}\int_K g_I^2 = ||g_I||_2^2 \frac{1}{|J|} = x_I^2 \frac{|I|}{|J|}.$$

Having made these observations it is easy to show that the orthogonal projection P defined by

$$P(f) = \sum_I \Big\langle f, \frac{g_I}{||g_I||_2}\Big\rangle \frac{g_I}{||g_I||_2}$$

is bounded in BMO. To prove boundedness of P, we fix $f \in$ BMO and let $g = P(f)$. Then we fix a dyadic interval J and $K \in \mathcal{B}_J$. We calculate using the previous observation,

$$\begin{aligned}\frac{1}{|K|}\int_K \Big| g - \frac{1}{|K|}\int_K g\Big|^2 &= \sum_{I\subseteq J} \Big\langle f, \frac{g_I}{||g_I||_2}\Big\rangle^2 \frac{1}{|K|}\int_K \frac{g_I^2}{||g_I||_2^2}\\ &= \Big\langle f, \frac{g_J}{||g_J||_2}\Big\rangle^2 \frac{1}{||g_J||_2^2} + \sum_{I\subset J}\Big\langle f, \frac{g_I}{||g_I||_2}\Big\rangle^2 \frac{1}{|J|}.\end{aligned}$$

We continue estimating the first summand of the above expression. Note that $||g_J||_2^2 = |B_J|$, and that the Haar support of g_J is the collection $\mathcal{B}_J$. Hence by

Bessel's inequality and Cauchy Schwartz we estimate as follows:

$$\begin{aligned}\left\langle f, \frac{g_J}{||g_J||_2}\right\rangle^2 \frac{1}{||g_J||_2^2} =&\left[\frac{1}{|B_J|}\sum_{K\in\mathcal{B}_J}\langle f,h_K\rangle\right]^2\frac{1}{|B_J|}\\ \leq&\frac{1}{|B_J|}\sum_{K\in\mathcal{B}_J}|K|\cdot\frac{1}{|B_J|}\sum_{K\in\mathcal{B}_J}\langle f,h_K\rangle^2|K|^{-1}\\ \leq& C||f||^2_{\mathrm{BMO}}.\end{aligned}$$

For the term on the right hand side we use Bessel's inequality. Recall that the supports of the functions g_I are contained in a set C_J with $|C_J| = |J|$. Recall also that the collection $\{C_I\}$ is a nested family of sets. Now we estimate using Bessel's inequality and the defintion of BMO:

$$\begin{aligned}\sum_{I\subset J}\left\langle f, \frac{g_I}{||g_I||_2}\right\rangle^2\frac{1}{|J|}\leq&\frac{1}{|J|}\sum_{K\subseteq C_J}\langle f,h_K\rangle^2|K|^{-1}\\ \leq&||f||^2_{\mathrm{BMO}}.\end{aligned}$$

Thus we showed that the orthogonal projection P is bounded on BMO.

With a similar calculation we evaluate now the norm in BMO of a linear combination $g = \sum a_I g_I$. To this end we fix a dyadic interval J and $K \in \mathcal{B}_J$. Then we calculate using the previous observation,

$$\begin{aligned}\frac{1}{|K|}\int_K\left|g-\frac{1}{|K|}\int_K g\right|^2 =&\sum_{I\subseteq J}\frac{1}{|K|}\int_K a_I^2g_I^2\\ =&a_J^2+\sum_{I\subset J}a_I^2x_I^2\frac{|I|}{|J|}.\end{aligned}$$

From this identity we deduce easily the following equivalence of norms,

$$||g||_{\mathrm{BMO}} \sim ||\sum a_I x_I h_I||_{\mathrm{BMO}} + \sup|a_I|.$$

Summing up, we showed that the weak *-closed linear span of $\{x_I\cdot h_I \oplus e_I\}$ in $\mathrm{BMO}\oplus\ell^\infty$ is isomorphic to a complemented subspace of BMO, namely to the weak *-closed linear span of $\{g_I\}$. ∎

THEOREM 2: *For every choice of scalars $x_I \in R$ with $|x_I| \leq 1$ the resulting space $\mathcal{S}^\infty$ is isomorphic to* BMO *or to ℓ^∞. If $\sum x_I h_I \in$ BMO then $\mathcal{S}^\infty$ is isomorphic to ℓ^∞, and if $\sum x_I h_I \notin$ BMO then $\mathcal{S}^\infty$ is isomorphic to* BMO.

Before we start the proof we recall convenient notation and useful convention. For a dyadic interval K we set

$$Q(K) = \{J \subseteq K : J \text{ is dyadic}\}.$$

Next we recall when a collection of dyadic intervals $\mathcal{H}_K$ is called a **block of intervals**. First we demand that $K \in \mathcal{H}_K$, and that $\mathcal{H}_K \subseteq Q(K)$. Second, if $I \in \mathcal{H}_K$, and if J is a dyadic interval satisfying $K \subseteq J \subseteq I$, then we have that $J \in \mathcal{H}_K$. Now we turn to the proof of Theorem 2.

Proof: We begin by performing a stopping time argument on the function $f = \sum x_I h_I$. For the unit interval $[0,1]$ we define $\mathcal{H}_{[0,1]}$ to be the largest block of dyadic intervals satisfying $[0,1] \in \mathcal{H}_{[0,1]}$ and

$$\sum_{L \in \mathcal{H}_{[0,1]}} x_L^2 h_L^2 \leq 2.$$

The maximality condition in the definition of $\mathcal{H}_{[0,1]}$ and the fact that $|x_I| \leq 1$ imply the following lower estimate. If $J \in Q([0,1]) \setminus \mathcal{H}_{[0,1]}$ then

$$\sum_{L \in \mathcal{H}_{[0,1]}} x_L^2 h_L^2 \geq 1 \text{ on the interval } J.$$

Next let K be a maximal interval in the collection $Q([0,1]) \setminus \mathcal{H}_{[0,1]}$. Then define $\mathcal{H}_K$ to be the maximal block of dyadic intervals contained in $Q(K)$ such that

$$\sum_{L \in \mathcal{H}_K} x_L^2 h_L^2 \leq 2.$$

Note that if $J \in Q(K) \setminus \mathcal{H}_K$ then

$$\sum_{L \in \mathcal{H}_K} x_L^2 h_L^2 \geq 1 \quad \text{on the interval } J.$$

This processs defines a decomposition of the dyadic intervals into a family of blocks $\{\mathcal{H}_K : K \in \mathcal{E}\}$. In this way we also obtain a decomposition of the function f into pieces $f_K = \sum_{L \in \mathcal{H}_K} x_L h_L$. This completes the first part of the proof.

Suppose now that $f = \sum x_I h_I \in \text{BMO}$. Let $\{a_I\}$ be a given sequence of scalars. Then we have the following upper bound,

$$\begin{aligned} \left\| \sum a_I x_I h_I \right\|_{\text{BMO}} &\leq \sup |a_I| \cdot \left\| \sum x_I h_I \right\|_{\text{BMO}} \\ &\leq \sup |a_I| \cdot \|f\|_{\text{BMO}}. \end{aligned}$$

Inserting this estimate in the equation defining the norm of $\mathcal{S}^\infty$ shows that $\mathcal{S}^\infty$ is isomorphic to ℓ^∞ when $f \in \text{BMO}$.

Now we turn to the last part of the proof examining consequences of the fact that $f \notin \text{BMO}$. We first observe that if $f \notin \text{BMO}$ then the index $\mathcal{E}$ satisfies,

$$\sup_I \frac{1}{|I|} \sum_{J \in \mathcal{E}, J \subseteq I} |J| = \infty.$$

The condensation lemma in [CG] implies that for any $n \in N$, there exists a dyadic interval $A \in \mathcal{E}$ such that

$$|G_n(A,\mathcal{E})| \geq |A|(1-8^{-n}),$$

where $G_n(A,\mathcal{E}) \subseteq A$ is the set of points $x \in A$, which are contained in n or more different intervals $J \in \mathcal{E} \cap Q(A)$.

Now let $\mathcal{X}_{[0,1]} = \{A\}$ be the collection containg just the interval A. Let J_1 be the left half of $[0,1]$ and let J_2 be the right half of $[0,1]$. We also let A_1 be the left half of A and let A_2 be the right half of A. Then define $\mathcal{X}_{J_i}$ to be the collection of maximal intervals in $\mathcal{E}$, which are contained in J_i. We call this the first step of the Gamlen–Gaudet construction. Next let K be any interval in $\mathcal{X}_{J_i}$. Let K_1 be the left half of K and let K_2 be the right half of K. Then let $\mathcal{X}(K_j)$ be the collection of maximal intervals in $\mathcal{E}$, which are contained in K_j. We let J_{i1} be the left half of J_i and we let J_{i2} be the right half of J_i. We will now take the union and put

$$\mathcal{X}_{J_{ij}} = \bigcup_{K \in \mathcal{X}_{J_i}} \mathcal{X}(K_j).$$

This is called the second step of the Gamlen–Gaudet construction. We repeat the basic step of the Gamlen–Gaudet construction at n times. With the assertion of the condensation lemma we obtain collections of pairwise disjoint dyadic intervals $\mathcal{X}_J \subseteq \mathcal{E}$, for $|J| \geq 2^{-n}$, satisfying the following properties (see [M]).

1. For $I \subseteq J$ let X_I be the pointset covered by the collection $\mathcal{X}_I$. Then for every $K \in \mathcal{X}_J$,

$$(1/2)|K| \cdot |I| \leq |J| \cdot |X_I \cap K| \leq 2|K| \cdot |I|.$$

2. If $X_I \cap X_J \neq \emptyset$, then either $X_I \subseteq X_J$ or $X_J \subseteq X_I$.
3. If $X_J \subseteq X_I$, then $J \subseteq I$.
4. $|A| \cdot |J|/2 \leq |X_J| \leq 2|A| \cdot |J|$.

Now define

$$G_J = \sum_{K \in \mathcal{X}_J} \sum_{L \in \mathcal{H}_K} g_L,$$

and also

$$H_J = \sum_{K \in \mathcal{X}_J} \sum_{L \in \mathcal{H}_K} x_L h_L.$$

Recall that we used $\{e_L\}$ indexed by dyadic intervals to denote the unit vector basis of ℓ^∞. Then put

$$F_J = \sum_{K \in \mathcal{X}_J} \sum_{L \in \mathcal{H}_K} e_L.$$

In the proof of the following statements the systems $\{H_J\}$ and $\{F_J\}$ play the role of auxiliary tools: First, we claim that in BMO the system $\{G_J : |J| \geq 2^{-n}\}$ is equivalent to the Haar basis $\{h_J : |J| \geq 2^{-n}\}$. And second, the orthogonal projection Q defined by

$$Q(y) = \sum_{\{I:|I|\geq 2^{-n}\}} \left\langle y, \frac{G_I}{||G_I||_2} \right\rangle \frac{G_I}{||G_I||_2}$$

satisfies

$$||Q(y)||_{\text{BMO}} \leq 4||y||_{\text{BMO}},$$

for $y \in \operatorname{span}\{g_J\}$.

We begin by showing the first claim doing the calculations with the system $\{H_J\}$. We fix J and let J_1 be the left half of J and let J_2 be the right half of J. Note that the square function $S^2(H_J) = \sum_{K\in\mathcal{X}_J} \sum_{L\in\mathcal{H}_K} x_L^2 h_L^2$ satisfies the pointwise upper estimate $S^2(H_J) \leq 2$, and also the following lower estimates,

$$S^2(H_J) \geq 1 \quad \text{on the set } X_{J_1} \cup X_{J_1}.$$

From this it follows easily that

$$\left\| \sum a_J H_J \right\|_{\text{BMO}} \sim \left\| \sum a_J h_J \right\|_{\text{BMO}}.$$

Next we claim that

$$|| \sum a_J F_J ||_{\ell^\infty} = \sup |a_J|.$$

To see this fix two dyadic intervals $J \neq J'$. Then the corresponding index sets given by $\mathcal{I}_J = \bigcup_{K\in\mathcal{X}_J} \mathcal{H}_K$ and $\mathcal{I}_{J'} = \bigcup_{K\in\mathcal{X}_J} \mathcal{H}_K$ are disjoint collections of intervals. This proves the claim. With Theorem 1, we have now the following equivalences proving the first claim,

$$\begin{aligned} \left\| \sum a_J G_J \right\|_{\text{BMO}} &\sim \left\| \sum a_J H_J \right\|_{\text{BMO}} + \sup |a_I| \\ &\sim \left\| \sum a_J h_J \right\|_{\text{BMO}}. \end{aligned}$$

Now we give the BMO estimates showing the boundedness of the orthogonal projection Q defined on $\operatorname{span}\{g_I\}$. We let

$$y = \sum a_I g_I.$$

Then

$$\langle y, G_J \rangle = \sum_{K\in\mathcal{X}_J} \sum_{L\in\mathcal{H}_K} a_L x_L^2 |L| \quad \text{and} \quad ||G_J||_2^2 = \sum_{K\in\mathcal{X}_J} \sum_{L\in\mathcal{H}_K} x_L^2 |L|.$$

Combining the above expressions with Hölder's inequality gives

$$\frac{\langle y, G_J\rangle^2}{||G_J||_2^2} \le \sum_{K\in\mathcal{X}_J}\sum_{L\in\mathcal{H}_K} a_L^2 x_L^2 |L|.$$

Recall that $\mathcal{X}_J$ consists of pairwise disjoint intervals, hence

$$|X_J| = \sum_{K\in\mathcal{X}_J} |K|.$$

For $K \in \mathcal{X}_J$, let $f_K = \sum_{L\in\mathcal{H}_K} x_L h_L$. Then the square function $S^2(f_K) = \sum_{L\in\mathcal{H}_K} x_L^2 h_L^2$ satisfies

$$\sum_{L\in\mathcal{H}_K} x_L^2|L| = \int_K S^2(f_K).$$

Next, by the stopping time construction and the condensation lemma we obtain

$$\frac{|K|}{2} \le \int_K S^2(f_K) \le 2|K|.$$

Taking the sum over $K \in \mathcal{X}_J$ gives

$$\frac{|X_J|}{2} \le \sum_{K\in\mathcal{X}_J}\int_K S^2(f_K) \le 2|X_J|.$$

Now observe that for J fixed and for I strictly contained in J, and $K \in \mathcal{X}_J$, we have the identities

$$\frac{1}{|K|}\int_K G_I^2 = \frac{||G_I||_2^2}{|X_J|} = \frac{|X_I|}{|X_J|}.$$

With the information collected so far we are now deriving estimates for $||Q(y)||_{\mathrm{BMO}}$. We fix J and $K \in \mathcal{X}_J$; then we compute obtaining global estimates,

$$\begin{aligned}\frac{1}{|K|}\int_K |Q(y) - \frac{1}{|K|}\int_K Q(y)|^2 &\le \frac{\langle y, G_J\rangle^2}{||G_J||_2^4} + \sum_{I\subset J}\Big\langle y, \frac{G_I}{||G_I||_2}\Big\rangle^2 \frac{1}{|K|}\int_K \frac{G_I^2}{||G_I||_2^2}\\ &\le \sup|a_I|^2 + \frac{1}{|B_J|}\sum_{I\subset J}\sum_{L\in\mathcal{X}_I} a_L^2 x_L^2|L|\\ &\le \sup|a_I|^2 + \Big\|\sum a_L x_L h_L\Big\|_{\mathrm{BMO}}^2\\ &\le \Big\|\sum a_L g_L\Big\|_{\mathrm{BMO}}^2.\end{aligned}$$

Finally, we point out that by the stopping time process we have the following local estimates. For any interval $L \in \mathcal{H}_K$ with $K \in \mathcal{X}_I$,

$$\frac{1}{|L|}\int_L |G_I - \frac{1}{|L|}\int_L G_I|^2 \leq C.$$

Combining the local estimate and the global estimates above shows that

$$||Q(y)||_{\mathrm{BMO}} \leq C||y||_{\mathrm{BMO}}.$$

Recall that BMO_n denotes the finite dimensional subspace of BMO which is spanned by the Haar functions $\{h_I : |I| \geq 2^{-n}\}$. So far we showed that for each $n \in N$ there is a well-complemented copy of BMO_n in $\mathrm{span}\{g_I\}$. We obtained these copies using only finite linear combinations of Haar functions. Therefore, we actually showed that the infinte direct sum $(\sum \mathrm{BMO}_n)_\infty$ is isomorphic to a complemented subspace of $(w* - \mathrm{span}\{g_I\}, ||\cdot||_{\mathrm{BMO}})$. (Here $w^* -$ span denotes the weak-* closure of the linear span.)

There exists a theorem of P. Wojtaszczyk to the effect that $(\sum \mathrm{BMO}_n)_\infty$ is isomorphic to BMO. (See [W, Theorem III.E.18].) Hence, by Theorem 1 and the Pelczynski decomposition, the space $(w^* - \mathrm{span}\{g_I\}, ||\cdot||_{\mathrm{BMO}})$ is isomorphic to BMO when $\sum x_I h_I \notin \mathrm{BMO}$. ∎

3. Extension to $L^p, 2 \leq p < \infty$

Recall that in section 2 we showed that the orthogonal projection onto the weak-* closed linear span of $\{g_I\}$, given by

$$P(f) = \sum_I \left\langle f, \frac{g_I}{||g_I||_2}\right\rangle \frac{g_I}{||g_I||_2},$$

is a bounded operator in BMO. It is also an operator of norm one in L^2. Thus by interpolation between BMO and L^2 and duality the orthogonal projection P is bounded on L^p, for $1 < p < \infty$. Hence the norm closed linear span of the system $\{g_I\}$ is a complemented subspace of L^p, for $1 < p < \infty$. Now we will restrict our attention to the range $2 \leq p < \infty$, since for these values of p we obtain considerably more information about the system of functions $\{g_I/||g_I||_p\}$.

THEOREM 3: *In $L^p, \infty > p \geq 2$, the system $g_I/||g_I||_p$, is equivalent to the system*

$$x_I^{1-2/p}\frac{h_I}{||h_I||_p} \oplus e_I,$$

in $L^p \oplus \ell^p$, where $\{e_I\}$ is the unit vector basis in ℓ^p and $\{h_I\}$ is the L^∞ normalized Haar system.

Proof: Let $a_I \in R$ be a given sequence of coefficients. We will show that then

$$\left\| \sum a_I \frac{g_I}{\|g_I\|_p} \right\|_p^p \sim \left\| \sum a_I x_I^{1-2/p} \frac{h_I}{\|h_I\|_p} \right\|_p^p + \sum |a_I|^p.$$

To this end we fix $n \in N$, and let I be a dyadic interval of length 2^{-n}. Then consider $K \in \mathcal{C}_I$. We let K_1 be the left half of K and K_2 be the right half of K. This gives the following collection of dyadic intervals,

$$\mathcal{H}_n = \bigcup \{K_1, K_2 : K \in \mathcal{C}_I\},$$

where the union is taken over all dyadic intervals I with $|I| = 2^{-n}$. Note that $\mathcal{H}_n$ is a collection of pairwise disjoint dyadic intervals covering the unit interval. Finally, let $\mathcal{F}_n$ be the σ-algebra generated by $\mathcal{H}_n$. Next write

$$d_n = \sum_{\{I:|I|=2^{-n}\}} a_I \frac{g_I}{\|g_I\|_p}.$$

The sequence $\{d_n\}$ is a martingale difference sequence with respect to the σ-algebras $\{\mathcal{F}_n\}$: That is, $E(d_n|\mathcal{F}_{n-1}) = 0$ and $E(d_n|\mathcal{F}_n) = d_n$. Next, we proceed by invoking a theorem of Burkholder to obtain

$$\left\| \sum d_n \right\|_p^p \sim \left\| \left(\sum E(d_n^2|\mathcal{F}_{n-1}) \right)^{1/2} \right\|_p^p + \sum \|d_n\|_p^p,$$

where the above equivalence depends on p. (See [B] and also [JMST] where several related applications of Burkholder's theorem are presented.) Now we are going to evaluate separately the two expressions on the right hand side of Burkholder's theorem. We begin with $\|d_n\|_p^p$. Recall that $I \cap J = \emptyset$ implies that $\operatorname{supp} g_I \cap \operatorname{supp} g_I = \emptyset$. Hence

$$\|d_n\|_p^p = \sum_{\{I:|I|=2^{-n}\}} |a_I|^p.$$

Next recall that

$$\|g_I\|_p^p = |\operatorname{supp} g_I| = x_I^2 \cdot |I|.$$

This gives that

$$E(d_n^2|\mathcal{F}_{n-1}) = \sum_{\{I:|I|=2^{-n}\}} a_I^2 E(g_I^2|\mathcal{F}_{n-1}) \frac{1}{x_I^{4/p} \cdot |I|^{2/p}}.$$

Now we seek an expression for $E(g_I^2|\mathcal{F}_{n-1})$. Note that for the dyadic interval I, with $|I| = 2^{-n}$, there exists a uniquely determined dyadic interval I^* with $|I^*| = 2^{-n+1}$, which contains I. Next we observe that by construction that we have

$$E(g_I^2|\mathcal{F}_{n-1}) = \frac{1}{2}x_I^2 1_{C_{I^*}}.$$

Inserting in the previous identity gives

$$\left\|\left(\sum_n E(d_n^2|\mathcal{F}_{n-1})\right)^{1/2}\right\|_p^p \sim \left\|\left(\sum_n \sum_{\{I:|I|=2^{-n}\}} a_I^2 x_I^{2-4/p} \frac{1_{C_{I^*}}}{|I|^{2/p}}\right)^{1/2}\right\|_p^p.$$

Finally, we use the tree structure of the sets $\{C_{I^*}\}$ and the square function norm on L^p to conclude that

$$\left\|\left(\sum_n \sum_{\{I:|I|=2^{-n}\}} a_I^2 x_I^{2-4/p} \frac{1_{C_{I^*}}}{|I|^{2/p}}\right)^{1/2}\right\|_p^p \sim \left\|\sum a_I x_I^{1-2/p} \frac{h_I}{\|h_I\|_p}\right\|_p^p.$$

Summing up we showed that

$$\left\|\sum a_I \frac{g_I}{\|g_I\|_p}\right\|_p^p \sim \left\|\sum a_I x_I^{1-2/p} \frac{h_I}{\|h_I\|_p}\right\|_p^p + \sum |a_I|^p,$$

as claimed. ∎

We identify $\mathcal{S}^p$ with the closed subspace of $L^p \oplus \ell^p$ that is spanned by the system

$$d_I = x_I^{1-2/p} \frac{h_I}{\|h_I\|_p} \oplus e_I,$$

since for a linear combination $\sum a_I d_I$ the norm in $L^p \oplus \ell^p$ is calculated as follows,

$$\left\|\sum a_I d_I\right\|_{L^p \oplus \ell^p}^p = \left\|\sum a_I x_I^{1-2/p} \frac{h_I}{\|h_I\|_p}\right\|_p^p + \sum |a_I|^p.$$

With the remark preceding the formulation of Theorem 3 we obtained that $\mathcal{S}^p$ is isomorphic to a complemented subspace of L^p. In that way we recoverd the result of D. Kleper and G. Schechtman. Now we continue classifying the isomorphic types of these spaces. What follows is a close examination of the stopping time decomposition $\{\mathcal{H}_K : K \in \mathcal{E}\}$, defined in the proof of Theorem 2. We use the notation $\limsup \mathcal{E}$ to denote the set of all points that are contained in infinitely many intervals of the collection $\mathcal{E}$.

THEOREM 4: *Let $x_I \in R$ be a sequence with $|x_I| \leq 1$. Then the infinite dimensional space $\mathcal{S}^p$ is either isomorphic to L^p or to ℓ^p. If $|\limsup \mathcal{E}| > 0$ then $\mathcal{S}^p$ is isomorphic to L^p, and if $|\limsup \mathcal{E}| = 0$ then $\mathcal{S}^p$ is isomorphic to ℓ^p.*

Proof: We begin with the proof of the second assertion. We show that if $|\limsup \mathcal{E}| = 0$, then the identity map $\mathrm{Id}\colon (\mathrm{span}\{g_I\}, ||\cdot||_p) \to L^p$ factors through ℓ^p. Here we rely on a theorem of W. B. Johnson asserting that such a factorization exists when the identity operator with range in L^2,

$$\mathrm{Id}\colon (\mathrm{span}\{g_I\}, ||\cdot||_p) \to L^p \to L^2,$$

is a *compact* operator. (See [J].)

Using that $|\limsup \mathcal{E}| = 0$ we will now verify that W. B. Johnson's criterion is satisfied. Let $R_n\colon\ L^2 \to L^2$ be the orthogonal projection onto $\mathrm{span}\{h_K : |K| \leq 2^{-n}\}$. Let

$$y = \sum a_I g_I,$$

with $||y||_p < \infty$. We compare the ratio of $||R_n(y)||_2$ to $||y||_p$ as follows. We consider the support set of $R_n(y)$ and put

$$F_n = \mathrm{supp}\, R_n(y).$$

Then a moment's reflection shows that

$$\lim_{n\to\infty} |F_n| \leq |\limsup \mathcal{E}| = 0.$$

Next we apply Hölder's inequality, and the fact that the basis constant of the Haar system in L^p equals one. This gives the following estimates,

$$\begin{aligned} ||R_n(y)||_2 \leq& ||R_n(y)||_p \cdot |F_n|^{1/2-1/p} \\ \leq& ||y||_p \cdot |F_n|^{1/2-1/p}. \end{aligned}$$

Summing up we proved that for $y = \sum a_I g_I$,

$$\lim_{n\to\infty} ||R_n(y)||_2 \leq \lim_{n\to\infty} ||y||_p \cdot |F_n|^{1/2-1/p} = 0.$$

This shows compactness for the identity map $\mathrm{Id}\colon (\mathrm{span}\{g_I\}, ||\cdot||_p) \to L^2$. Hence by the factorization theorem of W. Johnson, $\mathrm{Id}\colon (\mathrm{span}\{g_I\}, ||\cdot||_p) \to L^p$ factors through ℓ^p.

Now we prove that $(\mathrm{span}\{g_I\}, ||\cdot||_p)$ contains a complemented copy of L^p when the measure of $\limsup \mathcal{E}$ is strictly positive. We follow the proof in [M]. If $|\limsup \mathcal{E}| > 0$, then we may apply the basic step of the Gamlen–Gaudet construction infinitely many times. Having done this we obtain $\mathcal{X}_J \subseteq \mathcal{E}$, for J dyadic, so that the following conditions are satisfied.

1. Let $I \subseteq J$ and let X_I be the pointset covered by the collection $\mathcal{X}_I$. Then for $K \in \mathcal{B}_J$,
$$(1/2)|K| \cdot |I| \leq |J| \cdot |X_I \cap K| \leq 2|K| \cdot |I|.$$
2. If $X_I \cap X_J \neq \emptyset$, then either $X_I \subseteq X_J$ or $X_J \subseteq X_I$.
3. If $X_J \subseteq X_I$, then $J \subseteq I$.
4. $|A| \cdot |J|/2 \leq |X_J| \leq 2|A| \cdot |J|$.

In this way we constructed a dyadic tree of infinte depth. Now define for any dyadic interval J,

$$G_J = \sum_{K \in \mathcal{X}_J} \sum_{L \in \mathcal{H}_K} g_L.$$

In the course of proving Theorem 2 we showed that the orthogonal projection Q defined by

$$Q(y) = \sum_I \left\langle y, \frac{G_I}{||G_I||_2} \right\rangle \frac{G_I}{||G_I||_2},$$

satisfies

$$||Q(y)||_{\mathrm{BMO}} \leq 4||y||_{\mathrm{BMO}},$$

for $y \in \operatorname{span}\{g_J\}$. Recall also that the orthogonal projection

$$P(f) = \sum_I \left\langle f, \frac{g_I}{||g_I||_2} \right\rangle \frac{g_I}{||g_I||_2}$$

is a bounded operator on BMO. Hence the composition defined by

$$S(g) = Q(P(g))$$

is a bounded operator from BMO to BMO. Clearly S is also an operator of norm one from L^2 to L^2. Hence by interpolation between BMO and L^2, we obtain boundedness on $L^p, 2 \leq p < \infty$,

$$||S(g)||_p = ||Q(P(g))||_p \leq C_p||g||_p.$$

Now we evaluate the boundedness of S on $\operatorname{span}\{g_I\}$. For $y \in \operatorname{span}\{g_I\}$ we have clearly that $P(y) = y$, hence $Q(y) = S(y)$. Summing up we showed that by interpolation between BMO and L^2,

$$||Q(y)||_{L^p} \leq C_p||y||_{L^p},$$

for $y \in \operatorname{span}\{g_J\}$. Hence $(\operatorname{span}\{G_J\}, ||\cdot||_p)$ is complemented in $(\operatorname{span}\{g_J\}, ||\cdot||_p)$.

We prove next that in $L^p, 2 \leq p < \infty$, the system $\{G_J\}$ is equivalent to the Haar system. This will show that L^p is isomorphic to a complemented subspace of $(\text{span}\{g_J\}, ||\cdot||_p)$, and by Theorem 3 that L^p is isomorphic to a complemented subspace of $\mathcal{S}^p$. Now fix a sequence of scalars, $\{a_J\}$, then put

$$y = \sum a_J G_J.$$

Let $f_n = E(y|\mathcal{F}_n)$ be the conditional expectation with respect to the sequence of σ-algebras $\mathcal{F}_n$ defined earlier in this section. With Burkholder's theorem we evaluate the norm of y in $L^p, p \geq 2$. We have

$$||y||_p^p \sim \left\| \left(\sum E((f_n - f_{n+1})^2 | \mathcal{F}_{n-1}) \right)^{1/2} \right\|_p^p + \sum ||f_n - f_{n+1}||_p^p,$$

where the above equivalence depends on p. Now we unwind the definition of G_J and obtain a pointwise identity for the integrand of the first expression appearing in Burkholder's theorem,

$$\sum_n E((f_n - f_{n+1})^2 | \mathcal{F}_{n-1}) = \sum_J a_J^2 \sum_{K \in \mathcal{X}_J} \sum_{L \in \mathcal{H}_K} x_L^2 1_{C_{L^*}}.$$

Now we evaluate the second term in Burkholder' s theorem. We have that

$$\begin{aligned} \sum ||f_n - f_{n+1}||_p^p &= \sum_J |a_J|^p \sum_{K \in \mathcal{X}_J} \sum_{L \in \mathcal{H}_K} x_L^2 |L| \\ &\sim \sum_J |a_J|^p |X_J| \\ &\sim \sum_J |a_J|^p |J|. \end{aligned}$$

Next, we observe that for any $t \in \limsup \mathcal{E}$, we have the following pointwise estimates,

$$\frac{1}{2} 1_{C_{K^*}}(t) \leq \sum_{L \in \mathcal{H}_K} x_L^2 1_{C_{L^*}}(t) \leq 2 1_{C_{K^*}}(t).$$

With these observations we obtain that

$$\left\| \left(\sum_J a_J^2 \sum_{K \in \mathcal{X}_J} \sum_{L \in \mathcal{H}_K} x_L^2 1_{C_{L^*}} \right)^{1/2} \right\|_p \sim \left\| \sum a_J h_J \right\|_p.$$

Finally, recall that for $p \geq 2$,

$$\left\| \sum a_J h_J \right\|_p^p \geq c \sum_J |a_J|^p |J|.$$

Inserting these estimates back into Burkholder's theorem gives the equivalence

$$||y||_p \sim \left\| \sum a_J h_J \right\|_p .$$

Summing up we showed that $\mathcal{S}^p$ contains a complemented copy of L^p when $|\limsup \mathcal{E}| > 0$. On the other hand, by Theorem 3 and its preceding remark we know that $\mathcal{S}^p$ is isomorphic to a complemented subspace of L^p. Hence, by Pelczynski decomposition, we showed that $\mathcal{S}^p$ is isomorphic to L^p when $|\limsup \mathcal{E}| > 0$. ∎

4. Complemented subspaces of VMO

VMO is the norm closure of the set of finite linear combinations of Haar functions in BMO. Defined that way VMO is a separable space. Hence the space VMO has stronger ties to the scale of L^p spaces than BMO. In the classification problems considered in this section these ties can be observed easily.

We define the separable version of $\mathcal{S}^\infty$ as follows. We say that a null sequence (a_I) belongs to $\mathcal{S}^0$ if $\sum a_I x_I h_I \in VMO$, and we put

$$||(a_I)||_{\mathcal{S}^0} = \left\| \sum x_I a_I h_I \right\|_{VMO} + \sup |a_I|.$$

The proof of Theorem 1 shows that $\mathcal{S}^0$ is a complemented subspace of VMO. We will now determine the isomorphic types of $\mathcal{S}^0$.

THEOREM 5: *For every sequence of scalars $\{x_I\}$ with $|x_I \leq 1$, the infinite dimensional space $\mathcal{S}^0$ is isomorphic either to c_0, to $(\sum \mathrm{BMO}_n)_0$, or to* VMO. *If $\sum x_I h_I \in$ BMO, then $\mathcal{S}^0$ is isomorphic to c_0. If $\sum x_I h_I \notin$ BMO, and $|\limsup \mathcal{E}| = 0$, then $\mathcal{S}^0$ is isomorphic to $(\sum \mathrm{BMO}_n)_0$; if $|\limsup \mathcal{E}| > 0$ then $\mathcal{S}^0$ is isomorphic to* VMO.

Proof: We merge the proofs of Theorems 2 and 4.

If $\sum x_I h_I \in$ BMO, then the proof of Theorem 2 shows that $\mathcal{S}^0$ is isomorphic to c_0.

If $\sum x_I h_I \notin$ BMO, then the proof of Theorem 2 shows that $\mathcal{S}^0$ contains a complemented subspace isomorphic to $(\sum \mathrm{BMO}_n)_0$. If $|\limsup \mathcal{E}| = 0$, then we apply the VMO version of Johnson's factorization proved in [MS]. Indeed, when $|\limsup \mathcal{E}| = 0$, we observe easily that the identity operator with range in L^2, $\mathrm{Id}\colon (\mathrm{span}\{g_I\}, ||\cdot||_{\mathrm{VMO}}) \to \mathrm{VMO} \to L^2$, is a *compact* operator. The factorization theorem in [MS] asserts that then the identity operator with range in VMO, $\mathrm{Id}\colon (\mathrm{span}\{g_I\}, \|\cdot\|_{\mathrm{VMO}}) \to \mathrm{VMO}$, factors through $(\sum \mathrm{BMO}_n)_0$. Hence, by the Pelczynski decomposition principle, $\mathcal{S}^0$ is isomorphic to $(\sum \mathrm{BMO}_n)_0$, when $|\limsup \mathcal{E}| = 0$ and $\sum x_I h_I \notin \mathrm{BMO}$.

Finally, let $|\limsup \mathcal{E}| > 0$. Then we construct the system $\{G_J\}$ as in the proof of Theorem 4. The estimates in the proof of Theorem 2 show that the orthogonal projection onto $\mathrm{span}\{G_J\}$ is a bounded operator on VMO. Moreover, the Haar basis is equivalent to the system $\{G_J\}$ in VMO. Hence $(\mathrm{span}\{g_I\}, \|\cdot\|_{\mathrm{VMO}})$ contains a complemented copy of VMO, namely $(\mathrm{span}\{G_J\}, \|\cdot\|_{\mathrm{VMO}})$. By Pelczynski decomposition, we have proved that $\mathcal{S}^0$ is isomorphic to VMO. ∎

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